Fighting Boredom in Recommender Systems with Linear Reinforcement Learning

Romain WARLOP ROMAIN@FIFTY-FIVE.COM
fifty-five, Paris, France
SequeL Team, Inria Lille, France

Alessandro LAZARIC LAZARIC@FB.COM
Facebook AI Research, Paris, France

Jérémie MARY J.MARY@CRITEO.COM
Criteo AI Lab, Paris, France

Abstract

A common assumption in recommender systems (RS) is the existence of a *best fixed* recommendation strategy. Such strategy may be simple and work at the item level (e.g., in multi-armed bandit it is assumed one *best fixed* arm/item exists) or implement more sophisticated RS (e.g., the objective of A/B testing is to find the *best fixed* RS and execute it thereafter). We argue that this assumption is rarely verified in practice, as the recommendation process itself may impact the user's preferences. For instance, a user may get bored by a strategy, while she may gain interest again, if enough time passed since the last time that strategy was used. In this case, a better approach consists in alternating different solutions at the right frequency to fully exploit their potential. In this paper, we first cast the problem as a Markov decision process, where the rewards are a linear function of the recent history of actions, and we show that a policy considering the long-term influence of the recommendations may outperform both fixed-action and contextual greedy policies. We then introduce an extension of the UCRL algorithm (LinUCRL) to effectively balance exploration and exploitation in an unknown environment, and we derive a regret bound that is independent of the number of states. Finally, we empirically validate the model assumptions and the algorithm in a number of realistic scenarios.

1. Introduction

Consider a movie recommendation problem, where the recommender system (RS) selects the genre to suggest to a user. A basic strategy is to estimate user's preferences and then recommend movies of the preferred genres. While this strategy is sensible in the short term, it overlooks the *dynamics* of the user's preferences caused by the recommendation process. For instance, the user may get bored of the proposed genres and then reduce her ratings. This effect is due to the recommendation strategy itself and not by an actual evolution of user's preferences, as she would still like the same genres, if only they were not proposed so often.

The existence of an optimal *fixed* strategy is often assumed in RS using, e.g., matrix factorization to estimate users' ratings and the best (fixed) item/genre (Koren et al., 2009). Similarly, multi-armed bandit (MAB) algorithms (Bubeck and Cesa-Bianchi, 2012) effectively trade off exploration and exploitation in unknown environments, but still assume that rewards are independent from the sequence of arms selected over time and they try to select the (fixed) optimal arm as often as possible. Even when comparing more sophisticated recommendation strategies, as in A/B testing, we implicitly assume that once the better option (either A or B) is found, it should be constantly executed, thus ignoring how its performance may deteriorate if used too often. An alternative approach is to estimate the *state* of the user (e.g., her level of boredom) as a function of the movies recently watched and estimate how her preferences are affected by that. We could then learn a *contextual* strategy that recommends the best genre *depending* on the actual state of the user (e.g., using LinUCB (Li et al., 2010)). While this could partially address the previous issue, we argue that in practice it may not be satisfactory. As the preferences depend on the sequence of recommendations, a successful strategy should "drive" the user's state in the most favorable condition to gain as much reward as possible in the long term, instead selecting the best "instantaneous" action at each step. Consider a user with preferences *1) action*, *2) drama*, *3) comedy*. After showing a few *action* and *drama* movies, the user may get bored. A greedy contextual strategy would then move to recommending *comedy*, but as soon as it estimates that *action* or *drama* are better again (i.e., their potential value reverts to its initial value as they are not watched), it would immediately switch back to them. On the other hand, a more farsighted strategy may prefer to stick to *comedy* for a little longer to increase the preference of the user for *action* to its higher level and fully exploit its potential.

In this paper, we propose to use a reinforcement learning (RL) (Sutton and Barto, 1998) model to capture this dynamical structure, where the reward (e.g., the average rating of a genre) depends on a state that summarizes the effect of the recent recommendations on user's preferences. We introduce a novel learning algorithm that effectively trades off exploration and exploitation and we derive theoretical guarantees for it. Finally, we validate our model and algorithm in synthetic and real-data based environments.

Related Work. While in the MAB model, regret minimization (Auer, 2003) and best-arm identification algorithms (Jamieson and Talwalkar, 2016; Soare et al., 2014) have been often proposed to learn effective RS, they all rely on the assumption that one best fixed arm exists. (Heidari et al., 2016) study settings with time-varying rewards, where each time an arm is pulled, its reward decreases due to loss of interest, but, unlike our scenario, it never increases again, even if not selected for a long time. (Komiyama and Qin, 2014) also consider rewards that continuously decrease over time whether the arm is selected or not (e.g., modeling novelty effects, where new products naturally loose interest over time). This model fits into the more general case of restless bandit (e.g., Filippi et al., 2010; Tekin and Liu, 2012; Ortner et al., 2014), where each arm has a partially observable internal state that evolves as a Markov chain *independently* from the arms selected over time. Time-varying preferences has also been widely studied in RS. (Tekin and Liu, 2012; Koren, 2009) consider a time-dependent bias to capture seasonality and trends effect, but do not consider the effects on users' state. More related to our model is the setting proposed by (Shani et al., 2005), who consider an MDP-based RS at the item level, where the next item reward depends on the previously k selected items. Working at the item level without any underlying model assumption prevents their algorithm from learning in large state spaces, as every single combination of k items should be considered (in their approach this is partially mitigated by state aggregation). Finally, they do not consider the exploration-exploitation trade-off and they directly solve an estimated MDP. This may lead to an overall linear regret, i.e., failing to learn the optimal policy. Our work is also related to the linear bandit model (Li et al., 2010; Abbasi-yadkori et al., 2011), where rewards are a linear function of a context and an unknown target vector. Despite producing context-dependent policies, this model does not consider the influence that the actions may have on the state and thus overlook the potential of long-term reward maximization.

2. Problem Formulation

We consider a finite set of actions $a \in \{1, \ldots, K\} = [K]$. Depending on the application, actions may correspond to simple items or complex RS. We define the state s_t at time t as the history of the last w actions, i.e., $s_t = (a_{t-1}, \cdots, a_{t-w})$, where for $w = 0$ the state reduces to the empty history. As described in the introduction, we expect the reward of an action a to depend on how often a has been recently selected (e.g., a user may get bored the more a RS is used). We introduce the recency function $\rho(s_t, a) = \sum_{\tau=1}^{w} \mathbb{1}\{a_{t-\tau} = a\}/\tau$, where the effect of an action fades as $1/\tau$, so that the recency is large if an action is often selected and it decreases as it is not selected for a while. We define the (expected) reward function associated to an action a in state s as

$$r(s_t, a) = \sum_{j=0}^{d} \theta^*_{a,j} \rho(s_t, a)^j = x_{s,a}^\mathsf{T} \theta^*_a, \tag{1}$$

where $x_{s,a} = [1,\ \rho(s,a),\ \cdots,\ \rho(s,a)^d] \in \mathbb{R}^{d+1}$ is the context vector associated to action a in state s and $\theta^*_a \in \mathbb{R}^{d+1}$ is an unknown vector. In practice, the reward observed when selecting a at s_t is $r_t = r(s_t, a) + \varepsilon_t$, with ε_t a zero-mean noise. For $d = 0$ or $w = 0$, this model reduces to the standard MAB setting, where $\theta^*_{a,0}$ is the expected reward of action a. Eq. 1 extends the MAB model by summing the "stationary" component $\theta^*_{a,0}$ to a polynomial function of the recency $\rho(s_t, a)$. While alternative and more complicated functions of s_t may be used to model the reward, in the next section we show that a small degree polynomial of the recency is rich enough to model real data.

The formulation in Eq. 1 may suggest that this is an instance of a linear bandit problem, where $x_{s_t,a}$ is the context for action a at time t and θ^*_a is the unknown vector. Nonetheless, in linear bandit the sequence of contexts $\{x_{s_t,a}\}_t$ is independent from the actions selected over time and the optimal action at time t is $a^*_t = \arg\max_{a \in [K]} x_{s_t,a}^\mathsf{T} \theta^*_a$,[1] while in our model, $x_{s_t,a}$ actually depends on the state s_t, that summarizes the last w actions. As a result, an optimal policy should take into account its effect on the state to maximize the long-term average reward. We thus introduce the deterministic Markov decision process (MDP) $M = \langle S, [K], f, r \rangle$ with state space S enumerating the possible sequences of w actions, action space $[K]$, noisy reward function in Eq. 1, and a deterministic transition function $f : S \times [K] \to S$ that simply drops the action selected w steps ago and appends the last action to the state. A policy $\pi : S \to [K]$ is evaluated according to its long-term average reward as $\eta^\pi = \lim_{n\to\infty} \mathbb{E}[1/n \sum_{t=1}^{n} r_t]$, where r_t is the (random) reward of state s_t and action $a_t = \pi(s_t)$. The optimal policy is thus $\pi^* = \arg\max_\pi \eta^\pi$, with optimal average reward $\eta^* = \eta^{\pi^*}$. While an explicit form for π^* cannot be obtained in general, an optimal policy

1. We will refer to this strategy as "greedy" policy thereafter.

Algorithm 1 The LinUCRL algorithm.

Init: Set $t = 0$, $T_a = 0$, $\widehat{\theta}_a = 0 \in \mathbb{R}^{d+1}, V_a = \lambda I$
for rounds $k = 1, 2, \cdots$ **do**
  Set $t_k = t$, $\nu_a = 0$
  Compute $\widehat{\theta}_a = V_a^{-1} X_a^\mathsf{T} R_a$
  Set optimistic reward $\widetilde{r}_k(s, a) = x_{s,a}^\mathsf{T} \widehat{\theta}_a + c_{t,a} \|x_{s,a}\|_{V_a^{-1}}$
  Compute optimal policy $\widetilde{\pi}_k$ for MDP $(S, [K], f, \widetilde{r}_t)$
  while $\forall a \in [K], T_a < \nu_a$ **do**
    Choose action $a_t = \widetilde{\pi}_k(s_t)$
    Observe reward r_t and next state s_{t+1}
    Update $X_{a_t} \leftarrow [X_{a_t}, x_{s_t,a_t}]$, $R_{a_t} \leftarrow [R_{a_t}, r_t], V_{a_t} \leftarrow V_{a_t} + x_{s_t,a_t} x_{s_t,a_t}^\mathsf{T}$
    Set $\nu_{a_t} \leftarrow \nu_{a_t} + 1$, $t \leftarrow t + 1$
  end while
  Set $T_a \leftarrow T_a + \nu_a, \forall a \in [K]$
end for

may select an action with suboptimal instantaneous reward (i.e., action $a_t = \pi(s_t)$ is s.t. $r(s_t, a_t) < \max_a r(s_t, a)$) so as to let other (potentially more rewarding) actions "recharge" and select them later on. This results into a policy that alternates actions with a fixed schedule (see Sec. 4 for more insights).[2] If the parameters θ_a^* were known, we could compute the optimal policy by using value iteration where a value function $u_0 \in \mathbb{R}^S$ is iteratively updated as

$$u_{i+1}(s) = \max_{a \in [K]} \Big[r(s, a) + u_i(f(s, a)) \Big], \tag{2}$$

and a nearly-optimal policy is obtained after n iterations as $\pi(s) = \max_{a \in [K]} [r(s, a) + u_n(f(s, a))]$. Alternatively, algorithms to compute the maximum reward cycle for deterministic MDPs could be used (see e.g., Karp, 1978; Dasdan et al., 1999). The objective of a learning algorithm is to approach the performance of the optimal policy as quickly as possible. This is measured by the *regret*, which compares the reward cumulated over T steps by a learning algorithm and by the optimal policy, i.e.,

$$\Delta(T) = T\eta^* - \sum_{t=1}^{T} r(s_t, a_t), \tag{3}$$

where (s_t, a_t) is the sequence of states and actions observed and selected by the algorithm.

3. Linear Upper-Confidence bound for Reinforcement Learning

The Learning Algorithm. LinUCRL directly builds on the UCRL algorithm Jaksch et al. (2010) and exploits the linear structure of the reward function and the deterministic and known transition function f. The core idea of LinUCRL is to construct confidence intervals on the reward function and apply the optimism-in-face-of-uncertainty principle to compute an optimistic policy. The structure of LinUCRL is illustrated in Alg. 1. Let us

2. In deterministic MDPs the optimal policy is a recurrent sequence of actions inducing a maximum-reward cycle over states.

consider an episode k starting at time t, LinUCRL first uses the current samples collected for each action a separately to compute an estimate $\widehat{\theta}_{t,a}$ by regularized least squares, i.e.,

$$\widehat{\theta}_{t,a} = \min_{\theta} \sum_{\tau<t:a_\tau=a} (x_{s_\tau,a}^\mathsf{T}\theta - r_\tau)^2 + \lambda\|\theta\|_2, \tag{4}$$

where $x_{s_\tau,a}$ is the context vector corresponding to state s_τ and r_τ is the (noisy) reward observed at time τ. Let be $R_{a,t}$ the vector of rewards obtained up to time t when a was executed and $X_{a,t}$ the feature matrix corresponding to the contexts observed so far, then $V_{t,a} = (X_{t,a}^\mathsf{T} X_{t,a} + \lambda I) \in \mathbb{R}^{(d+1)\times(d+1)}$ is the design matrix. The closed-form solution of the estimate is $\widehat{\theta}_{t,a} = V_{t,a}^{-1} X_{t,a}^\mathsf{T} R_{t,a}$, which gives an estimated reward function $\widehat{r}_t(s,a) = x_{s,a}^\mathsf{T}\widehat{\theta}_{t,a}$. Instead of computing the optimal policy according to the estimated reward, we compute the upper-confidence bound

$$\widetilde{r}_t(s,a) = \widehat{r}_t(s,a) + c_{t,a}\|x_{s,a}\|_{V_{t,a}^{-1}}, \tag{5}$$

where $c_{t,a}$ is a scaling factor whose explicit form is provided in Eq. 7. Since the transition function f is deterministic and known, we then simply apply the value iteration scheme in Eq. 2 to the MDP $\widetilde{M}_k = \langle S, [K], f, \widetilde{r}_k\rangle$ and compute the corresponding optimal (optimistic) policy $\widetilde{\pi}_k$. It is simple to verify that $(\widetilde{M}_k, \widetilde{\pi}_k)$ is the pair of MDP and policy that maximizes the average reward over all "plausible" MDPs that are within the confidence intervals over the reward function. More formally, let $\mathcal{M}_k = \{M = \langle S, [A], f, r\rangle,\ |r(s,a) - \widehat{r}_t(s,a)| \leq c_{t,a}\|x_{s,a}\|_{V_{t,a}^{-1}}, \forall s, a\}$, then with high probability we have that

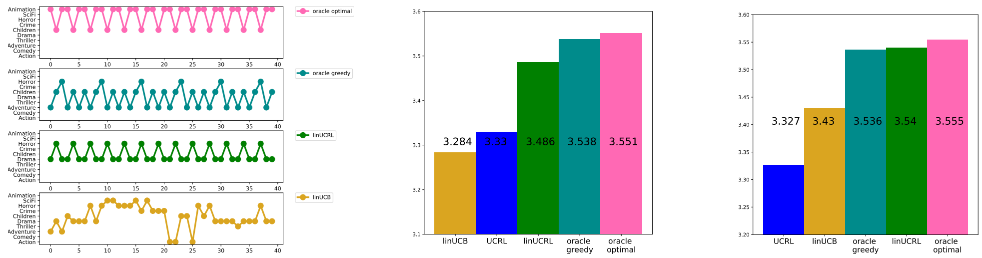
$$\eta^{\widetilde{\pi}_k}(\widetilde{M}_k) \geq \max_{\pi, M\in\mathcal{M}_k} \eta^\pi(M). \tag{6}$$

Finally, LinUCRL execute $\widetilde{\pi}_k$ until the number of samples for an action is doubled w.r.t. the beginning of the episode. The specific structure of the problem makes LinUCRL more efficient than UCRL, since each iteration of Eq. 2 has $O(dSK)$ computational complexity compared to $O(S^2K)$ of extended value iteration (used in UCRL) due to the randomness of the transitions and the optimism over f.

Theoretical Analysis. We prove that LinUCRL successfully exploits the structure of the problem to reduce its cumulative regret w.r.t. basic UCRL. We first make explicit the confidence interval in Eq. 5. Let assume that there exist (known) constants B and R such that $\|\theta_a^*\|_2 \leq B$ for all actions $a \in [K]$ and the noise is sub-Gaussian with parameter R. Let $\ell_w = \log(w) + 1$, where w is the length of the window in the state definition, and $L_w^2 = \frac{1-\ell_w^{d+1}}{1-\ell_w}$, where d is the degree of the polynomial describing the reward function. Then, we run LinUCRL with the scaling factor

$$c_{t,a} = R\sqrt{(d+1)\log\Big(Kt^\alpha\Big(1+\frac{T_{t,a}L_w^2}{\lambda}\Big)\Big)} + \lambda^{1/2}B \tag{7}$$

where $T_{t,a}$ is the number of samples collected from action a up to t. Then we can prove the following.



(a) Last 40 actions *(b)* Avg. rwd. at $T = 200$ *(c)* Avg. rwd. at the end

Figure 1: Results of learning experiment based on *movielens* dataset.

Theorem 1 *If* LinUCRL *runs with the scaling factor in Eq. 7 over* T *rounds, then its cumulative regret is*

$$\Delta(\text{LinUCRL}, T) \leq Kw \log_2\left(\frac{8T}{K}\right) + 2c_{\max}\sqrt{2KT(d+1)\log\left(1 + \frac{TL_w^2}{\lambda(d+1)}\right)}, \qquad (8)$$

where $c_{\max} = \max_{t,a} c_{t,a}$.

From this theorem we notice that the per-step regret Δ/T decreases to zero as $1/\sqrt{T}$, showing that as time increases, the reward approaches the optimal average reward. More insights on this theorem are available in the long paper version.

4. Experiments

In order to validate our model on real datasets, we need persistent information about a user identification number to follow the user through time and evaluate how preferences evolve over time in response to the recommendations. This also requires datasets where several RSs are used for the same user with different cadence and for which it is possible to associate a user-item feedback with the system that actually performed that recommendation. Unfortunately, these requirements make most of publicly available datasets not suitable for this validation. As a result, we propose to use dataset-based experiments to empirically validate our model and compare LinUCRL to existing baselines. We consider two different scenarios. *Movielens:* We derive model parameters from the *movielens* dataset and we compare the learning performance (i.e., cumulative reward) of LinUCRL to baseline algorithms. *Real-world data from A/B testing:* this dataset provides enough information to test our algorithm and although our model assumptions are no longer satisfied, we can still investigate how a long-term policy alternating A and B on the basis of past choices can outperform each solution individually.

Learning on *movielens* dataset. In order to overcome the difficulty of creating full complex RS and evaluate them on offline datasets, we focus on a relatively simple scenario where a RS directly recommends movies from one chosen genre. One strategy could be to apply a bandit algorithm to find the optimal genre and then always recommend movies of this genre. On the other hand, our algorithm tries to identify an optimal sequence of those genres to keep the user interested. The standard offline evaluation of a learning algorithm on historical data is to use a replay or counterfactual strategy Li et al. (2011); Swaminathan and Joachims (2015), which consists in updating the model whenever the learning algorithm takes the same action as in the logged data, and only update the state (but not the model) otherwise. In our case this replay strategy cannot be applied because the reward depends on the history of selected actions and we could not evaluate the reward of an action if the algorithm generated a sequence that is not available in the dataset (which is quite likely). Thus in order to compare the learning performance of LinUCRL to existing baselines, we use the *movielens100k* dataset to estimate the parameters of our model and construct the corresponding "simulator". Unlike a fully synthetic experiment, this gives a configuration which is "likely" to appear in practice, as the parameters are directly estimated from real data. We choose $K = 10$ actions corresponding to different genres of movies, and we set $d = 5$ and $w = 5$, which results into $K^w = 10^5$ states. We recall that w has a mild impact on the learning performance of LinUCRL as it does not need to repeatedly try the same action in each state (as UCRL) to be able to estimate its reward. This is also confirmed by the regret analysis that shows that the regret only depends on w in the lower-order logarithmic term of the regret. Given this number of states, UCRL would need at least one million iteration to observe each state 10 times which is dramatically too large for the application we consider. The parameters that describe the dependency of the reward function on the recency (i.e., $\theta^*_{j,a}$) are computed by using the ratings averaged over all users for each state encountered and for ten different genres in the dataset. The first component of the vectors θ^*_a is chosen to simulate different user's preferences and to create complex dynamics in the reward functions. The resulting parameters and reward functions are reported in appendix. Finally, the observed reward is obtained by adding a small random Gaussian noise to the linear function. In this setting, a constant strategy would always pull the comedy genre since it is the one with the highest "static" reward, while other genres are also highly rewarding and a suitable alternation between them may provide a much higher reward.

We compare LinUCRL to the following algorithms: *oracle optimal* (π^*), *oracle greedy* (greedy contextual policy), LinUCB Abbasi-yadkori et al. (2011) (learn the parameters using LinUCB for each action and select the one with largest instantaneous reward), UCRL Auer et al. (2009) (considering each action and state independently). The results are obtained by averaging 4 independent runs. Fig. 1*(*b-c) shows the average reward at $T = 200$ and after $T = 2000$ steps. We first notice that as in the previous experiment the oracle greedy policy is suboptimal compared to the optimal policy that maximizes the long-term reward. Despite the fact that UCRL targets this better performance, the learning process is very slow as the number of states is too large. Indeed this number of steps is lower than the number of states so UCRL did not have the chance to update its policy since in average no states has been visited twice. On the other hand, at early learning stages LinUCRL is already better than LinUCB, and its performance keeps improving until, at 2000 steps, it actually performs better than the oracle greedy strategy and it is close to the optimal policy.

Large scale *A/B* testing dataset. We also validate our approach on a real-world A/B testing dataset. We collected 15 days of click on ads history of a criteo's test, where users have been proposed two variations on the display denoted as A and B. Each display is actually the output of two real-world collaborative-filtering recommender strategies; precise information on how these algorithms are constructed is not relevant for our analysis. Unlike a classical A/B testing each unique user has been exposed to *both* A and B but with different frequencies. This dataset is formed of 350M tuples *(user id, timestamp, version, click)* and will be released publicly as soon as possible. Remark that the system is already heavily

Algorithm	on the T steps	on the last steps
only B	46.0%	46.0%
UCRL	46.5%	46.0%
LinUCRL	66.7%	75.8%
oracle greedy	61.3%	61.3%
oracle optimal	95.2%	95.2%

Table 1: Relative improvement over *only A* of learning experiment based on *large scale A/B testing* dataset.

optimized and that even a small improvement in the click-rate is very desirable. As in the *movielens* experiment, we do not have enough data to evaluate a learning algorithm on the historical events (not enough samples per state would be available), so we first compute a simulator based on the data and then run LinUCRL- that does not know the parameters of the simulator and must try to estimate them - and compare it to simple baselines. Unlike the previous experiment, we do not impose any linear assumption on the simulator (as in Eq. 1) and we compute the click probability for actions A and B independently in each state (we set $w = 10$, for a total of $2^{10} = 1024$ states) and whenever that state-action pair is executed we draw a Bernoulli with the corresponding probability. Using this simulator we compute oracle greedy and optimal policies and we compare LinUCB, LinUCRL, which is no longer able to learn the "true" model, since it does not satisfy the linear assumption, and UCRL, which may suffer from the large number of state but targets a model with potentially better performance (as it can correctly estimate the actual reward function and not just a linear approximation of it). We report the results (averaged over 5 runs) as a relative improvement over the worst fixed option (i.e., in this case A). Tab. 1 shows the average reward over $T = 2,000$ steps and of the learned policy at the end of the experiment. Despite the fact that the simulator does not satisfy our modeling assumptions, LinUCRL is still the most competitive algorithm as it achieves the best performance among the learning algorithms and it outperforms the oracle greedy policy.

5. Conclusion

We showed that estimating the influence of the recommendation strategy on the reward and computing a policy maximizing the long-term reward may significantly outperform fixed-action or greedy contextual policies. We introduced a novel learning algorithm, LinUCRL, to effectively learn such policy and we prove that its regret is much smaller than for standard reinforcement learning algorithms (UCRL). We validated our model and its usefulness on the *movielens* dataset and on a novel A/B testing dataset. Our results illustrate how the optimal policy effectively alternates between different options, in order to keep the interest of the users as high as possible. Furthermore, we compared LinUCRL to a series of learning baselines on simulators satisfying our linearity assumptions (*movielens*) or not (A/B testing). A venue for future work is to extend the current model to take into consideration correlations between actions. Furthermore, given its speed of convergence, it could be interesting to run a different instance of LinUCRL per user - or group of users - in order to offer personalized "boredom" curves. Finally, using different models of the reward as a function of the recency (e.g., logistic regression) could be used in case of binary rewards.

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Appendix A. Proof of Theorem 1

Proof. In order to prove Thm. 1, we first need the following proposition about the confidence intervals used in computing the optimistic reward $\widetilde{r}(s,a)$.

Proposition 2 *Let assume $\|\theta^*_a\|_2 \leq B$. If $\widehat{\theta}_{t,a}$ is computed as in Eq. 4 and $c_{t,a}$ is defined as in Eq. 7, then*

$$\mathbb{P}\left(r(s,a) \leq \widehat{r}(s,a) + c_{t,a}\|x_{s,a}\|_{V_{t,a}^{-1}}\right) \leq \frac{t^{-\alpha}}{K}. \tag{9}$$

Proof By definition of $\rho(s,a)$ we have $0 \leq \rho(s,a) \leq \sum_{\tau=1}^{w} \frac{1}{\tau} < \log(w)+1 \doteq \ell_w$. Thus $1 \leq \|x_{s,a}\|_2^2 \leq \sum_{j=0}^{d} \ell_w^j = \frac{1-\ell_w^{d+1}}{1-\ell_w} = L_w^2$. Using Thm. 2 of Abbasi-yadkori et al. (2011), we have with probability $1-\delta$,

$$\|\widehat{\theta}_{t,a} - \theta^*_a\|_{V_{t,a}} \leq R\sqrt{(d+1)\log\left(\frac{1+T_{t,a}L_w^2/\lambda}{\delta}\right)} + \lambda^{1/2}B.$$

Thus for all $s \in S$ we have,

$$|r(s,a) - \widehat{r}(s,a)| = |x_{s,a}^\mathsf{T}\widehat{\theta}_{t,a} - x_{s,a}^\mathsf{T}\theta^*_a| \leq \|x_{s,a}\|_{V_{t,a}^{-1}}\|\widehat{\theta}_a - \theta^*_a\|_{V_{t,a}}. \tag{10}$$

Using $\delta = \frac{t^{-\alpha}}{K}$ concludes the proof. ■

An immediate result of Prop. 2 is that the estimated average reward of $\widetilde{\pi}_k$ in the optimistic MDP $\widetilde{M}_k$ is an upper-confidence bound on the optimal average reward, i.e., for any t (the probability follows by a union bound over actions)

$$\mathbb{P}(\eta^* > \eta^{\widetilde{\pi}_k}(\widetilde{M}_k)) \leq t^{-\alpha}. \tag{11}$$

We are now ready to prove the main result.
Proof [Proof of Thm. 1] We follow similar steps as in Jaksch et al. (2010). We split the regret over episodes as

$$\Delta(\mathcal{A},T) = \sum_{k=1}^{m}\sum_{t=t_k}^{t_{k+1}-1}(\eta^* - r(s_t,a_t)) = \sum_{k=1}^{m}\Delta_k. \tag{12}$$

Let $\mathcal{T}_{k,a} = \{t_k \leq t < t_{k+1} : a_t = a\}$ be the steps when action a is selected during episode k. We upper bound the per-episode regret as

$$\Delta_k = \sum_{a\in[K]}\sum_{t\in\mathcal{T}_{k,a}}\big(\eta^* - r(s_t,a)\big) \leq \sum_{t=t_k}^{t_{k+1}-1}\big(\widetilde{\eta}_k - \widetilde{r}_k(s_t,a)\big) + \sum_{a\in[K]}\sum_{t\in\mathcal{T}_{k,a}}\big(\widetilde{r}_k(s_t,a) - r(s_t,a)\big), \tag{13}$$

where the inequality directly follows from the event that $\widetilde{\eta}_k \geq \eta^*$ (Eq. 11) with probability $1-T^{-\alpha}$. Notice that the low-probability event of failing confidence intervals can be treated as in Jaksch et al. (2010).

We proceed by bounding the first term of Eq. 13. Unlike in the general online learning scenario, in our setting the transition function f is known and thus the regret incurred from bad estimates of the dynamics is reduced to zero. Furthermore, since we are dealing with deterministic MDPs, the optimal policy converges to a loop over states. When starting a new policy, we may start from a state outside its loop. Nonetheless, it is easy to verify that starting from any state s, it is always possible to reach any desired state s' in at most w steps (i.e., the size of the history window). As a result, within each episode k the difference between the cumulative reward ($\sum_t \widetilde{r}_k(s_t, a)$) and the (optimistic) average reward ($(t_{k+1} - t_k)\widetilde{\eta}_k$) in the loop never exceeds w. Furthermore, since episodes terminate when one action doubles its number of samples, using a similar proof as Jaksch et al. (2010), we have that the number of episodes is bounded as $m \leq K \log_2(\frac{8T}{K})$. As a result, the contribution of the first term of Eq. 13 to the overall regret is bounded as

$$\sum_{k=1}^{m} \sum_{t=t_k}^{t_{k+1}-1} (\widetilde{\eta}_k - \widetilde{r}_k(s_t, a)) \leq Kw \log_2\Big(\frac{8T}{K}\Big). \tag{14}$$

The second term in Eq. 13 refers to the (cumulative) reward estimation error and it can be decomposed as

$$|\widetilde{r}_k(s_t, a) - r(s_t, a)| \leq |\widetilde{r}_k(s_t, a) - \widehat{r}_k(s_t, a)| + |\widehat{r}_k(s_t, a) - r(s_t, a)|. \tag{15}$$

We can bound the cumulative sum of the second term as (similar for the first, since $\widetilde{r}_k$ belongs to the confidence interval of $\widehat{r}_k$ by construction)

$$\begin{aligned} \sum_{k=1}^{m} \sum_{a \in [K]} \sum_{t \in \mathcal{T}_{k,a}} |\widehat{r}_k(s_t, a) - r(s_t, a)| &\leq \sum_{k=1}^{m} \sum_{a \in [K]} \sum_{t \in \mathcal{T}_{k,a}} c_{t,a} \|x_{s_t,a}\|_{V_{a,t}^{-1}} && (16) \\ &\leq c_{\max} \sum_{a \in [K]} \sqrt{\sum_{k=1}^{m} \sum_{t \in \mathcal{T}_{k,a}} \|x_{s_t,a}\|^2_{V_{a,t}^{-1}}} \sqrt{T_a}, && (17) \end{aligned}$$

where the first inequality follows from Prop. 2 with probability $1 - T^{-\alpha}$, and T_a is the total number of times a has been selected at step T. Let $\mathcal{T}_a = \cup_k \mathcal{T}_{k,a}$, then using Lemma 11 of Abbasi-yadkori et al. (2011), we have

$$\sum_{t \in \mathcal{T}_a} \|x_{s_t,a}\|^2_{V_{t,a}^{-1}} \leq 2 \log \frac{\det(V_{T,a})}{\det(\lambda I)},$$

and by Lem. 10 of (Abbasi-yadkori et al., 2011), we have

$$\det(V_{t,a}) \leq (\lambda + tL_w^2/(d+1))^{d+1},$$

which leads to

$$\begin{aligned} \sum_{k=1}^{m} \sum_{a \in [K]} \sum_{t \in \mathcal{T}_{k,a}} |\widehat{r}_k(s_t, a) - r(s_t, a)| &\leq c_{\max} \sum_{a \in [K]} \sqrt{T_a} \sqrt{2(d+1) \log\Big(\frac{\lambda + tL_w^2}{\lambda(d+1)}\Big)} && (18) \\ &\leq c_{\max} \sqrt{2KT(d+1) \log\Big(\frac{\lambda + tL_w^2}{\lambda(d+1)}\Big)}. && (19) \end{aligned}$$

Bringing all the terms together gives the regret bound. ■

Discussion on the regret bound. We first notice that the per-step regret Δ/T decreases to zero as $1/\sqrt{T}$, showing that as time increases, the reward approaches the optimal average reward. Furthermore, by leveraging the specific structure of our problem, LINUCRL greatly improves the dependency on other elements characterizing the MDP. In the general MDP case, UCRL suffers from a regret $O(DS\sqrt{KT})$, where D is the diameter of the MDP, which in our case is equal to the history window w. In the regret bound of LINUCRL the dependency on the number of states (which is exponential in the history window $S = K^w$) disappears and it is replaced by the number of parameters $d + 1$ in the reward model. Furthermore, since the dynamics is deterministic and known, the only dependency on the diameter w is in a lower-order logarithmic term. This result suggests that we can take a large window w and a complex polynomial expression for the reward (i.e., large d) without compromising the overall regret. Let note that in MDPs, the worst-case regret lower bound also exhibits a $\sqrt{(T)}$ dependency (Jaksch et al. (2010)), so there is not much hope to improve it. The interesting part of these bounds is actually in the problem-specific terms. Furthermore, LINUCRL compares favorably with a linear bandit approach. First, η^* is in general much larger than the optimal average reward of a greedy policy selecting the best instantanous action at each step. Second, apart from the $\log(T)$ term, the regret is the same of a linear bandit algorithm (e.g., LINUCB). This means that LINUCRL approaches a better target performance η^* almost at the same speed as linear bandit algorithms reach a worse greedy policy. Finally, (Ortner, 2008) developed a specific instance of UCRL for deterministic MDPs, whose final regret is of order $O(\lambda A \log(T)/\Delta)$, where λ is the length of the largest simple cycle that can be generated in the MDP and Δ is the gap between the reward of the optimal and second-optimal policy. While the regret in this bound only scales as $O(\log T)$, in our setting λ can be as large as $S = K^w$, which is exponentially worse than the diameter w, and Δ can be arbitrarily small, thus making a $O(\sqrt{T})$ bound often preferable. We leave the integration of our linear reward assumption into the algorithm proposed by (Ortner, 2008) as future work.

Appendix B. Experiments Details

Genre	$\theta^*_{a,0}$	$\theta^*_{a,1}$	$\theta^*_{a,2}$	$\theta^*_{a,3}$	$\theta^*_{a,4}$	$\theta^*_{a,5}$
Action	3.1	0.54	-1.08	0.78	-0.22	0.02
Comedy	3.34	0.54	-1.08	0.78	-0.22	0.02
Adventure	3.51	0.86	-2.7	3.06	-1.46	0.24
Thriller	3.4	1.26	-2.9	2.76	-1.14	0.16
Drama	2.75	1.0	0.94	-1.86	0.94	-0.16
Children	3.52	0.1	0.0	-0.3	0.2	-0.04
Crime	3.37	0.32	1.12	-3.0	2.26	-0.54
Horror	3.54	-0.68	1.84	-2.04	0.82	-0.12
SciFi	3.3	0.64	-1.32	1.1	-0.38	0.02
Animation	3.4	1.38	-3.44	3.62	-1.62	0.24

Table 2: Reward parameters of each genre for the *movielens* experiment.

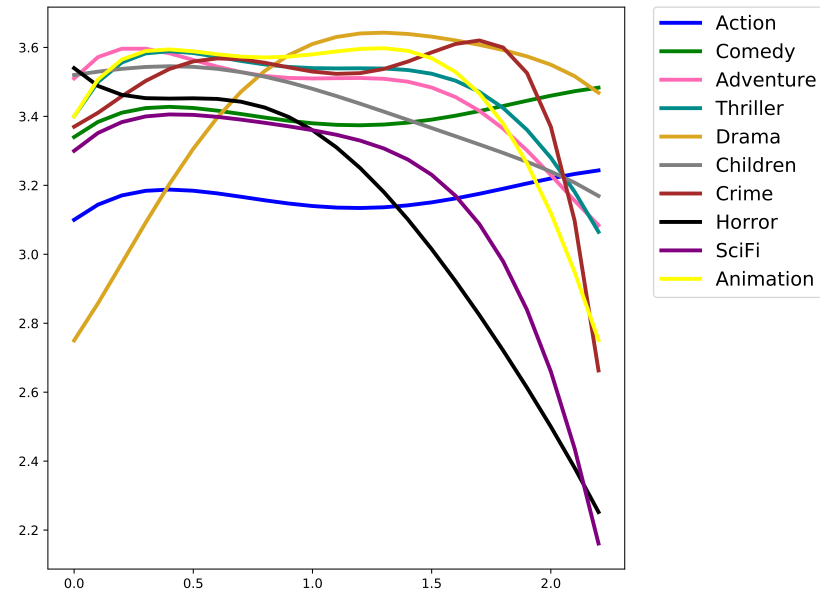

Figure 2: Reward functions constructed from the *movielens* dataset.

The parameters used in the MovieLens experiment are reported in Table 2, while the corresponding reward functions are shown in Fig. 2.