On Periodic Markov Decision Processes

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Outline

1. Markov Decision Processes
2. Periodic Markov Decision Processes
3. Approximate Dynamic Programming

Markov Decision Process (MDP)

(Puterman, 1994; Bertsekas & Tsitsiklis, 1996; Sutton & Barto, 1998)

Controlled and rewarded dynamical system:

$$x_0,\ a_0,\ r_0,\ x_1,\ a_1,\ r_1,\ x_2,\ a_2,\ r_2,\ x_3,\ \dots$$

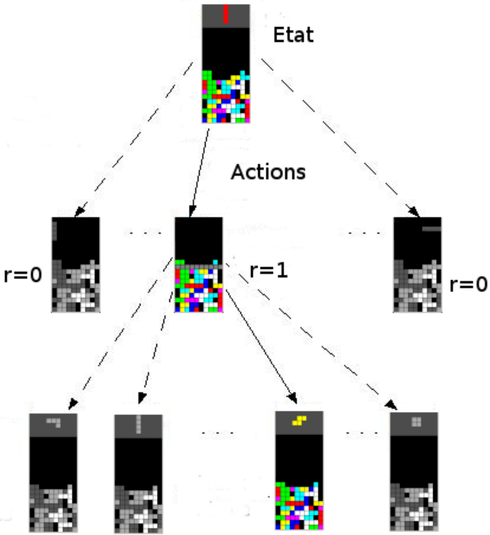
Markov Decision Process (MDP):

- X is the (countable) state space,
- A is the (countable) action space,
- $r : X \times A \to \mathbb{R}$ is the reward function, $\quad (r_t = r(x_t, a_t))$
- $p : X \times A \to \Delta_X$ is the transition kernel. $\quad (x_{t+1} \sim p(\cdot|x_t, a_t))$

Goal: Find a stationary deterministic policy $\pi : X \to A$ that maximizes the value $v_\pi(x)$ *for all* x:

$$v_\pi(x) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t \middle| x_0 = x,\ \{\forall t,\ a_t = \pi(x_t)\}\right]. \qquad (\gamma \in (0,1))$$

Illustration: Tetris



Bellman Equations/Operators

- For any policy π, v_π is the unique solution of the **Bellman equation**:

$$\forall x,\ v_\pi(x) = r(x,\pi(x)) + \gamma \sum_{y\in X} p(y|x,\pi(x)) v_\pi(y) \quad \Leftrightarrow \quad v_\pi = T_\pi v_\pi .$$

- The **optimal value** v_* is the unique solution of the **Bellman optimality equation**:

$$\forall x,\ v_*(x) = \max_{a\in A} \left(r(x,a) + \gamma \sum_{y\in X} p(y|x,a) v_*(y) \right) \quad \Leftrightarrow \quad v_* = T v_* .$$

- $T_\pi : \mathbb{R}^X \to \mathbb{R}^X$ and $T : \mathbb{R}^X \to \mathbb{R}^X$ are γ-contraction mappings w.r.t. the max norm $\|v\|_\infty = \max_s |v(s)|$.
- For any v, π is a **greedy policy** w.r.t. v , written $\pi = \mathcal{G}v$, iff

$$\forall x,\ \pi(x) \in \arg\max_{a\in A} \left(r(x,a) + \gamma \sum_{y\in X} p(y|x,a) v(y) \right) \quad \Leftrightarrow \quad T_\pi v = T v .$$

- $\pi_* = \mathcal{G} v_*$

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Dynamic Programming Algorithms

Value Iteration

$\pi_{k+1} \leftarrow \mathcal{G} v_k$
$v_{k+1} \leftarrow T v_k = T_{\pi_{k+1}} v_k$

Policy Iteration

$\pi_{k+1} \leftarrow \mathcal{G} v_k$
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Modified Policy Iteration (Puterman & Shin, 1978)

$\pi_{k+1} \leftarrow \mathcal{G} v_k$
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Asynchronous Dynamic Programming

Subsets: $X_1, X_2, \ldots, X_k, \ldots \qquad X_i \subset X$

At each iteration k,

- Either update the value on X_k

$$\begin{aligned} \pi_{k+1} &= \pi_k \\ v_{k+1}(x) &= \begin{cases} [T_{\pi_k} v_k](x) & \text{if } x \in X_k \\ v_k(x) & \text{otherwise} \end{cases} \end{aligned}$$

- or update the policy on X_k

$$\begin{aligned} \pi_{k+1}(x) &= \begin{cases} [\mathcal{G} v_k](x) & \text{if } x \in X_k \\ \pi_k(x) & \text{otherwise} \end{cases} \\ v_{k+1} &= v_k \end{aligned}$$

Convergence if all states are updated infinitely often (Bertsekas & Tsitsiklis, 1996).

Outline

ℓ-periodic MDP (Riis, 1965)

Let $\ell \geq 1$. Write $[t] = t \bmod \ell$.
At time t, the reward and the transition used are $r_{[t]}$ and $p_{[t]}$:

- $r_0, r_1, \ldots, r_{\ell-1}$, with $r_i : X \times A \to \mathbb{R}$
- $p_0, p_1, \ldots, p_{\ell-1}$, with $p_i : X \times A \to \Delta_X$

An ℓ-periodic MDP is an MDP on

$$X \times \{0, 1, \ldots, \ell - 1\} \quad = \quad X_0 \cup X_1 \cup \cdots \cup X_{\ell-1}$$

$\Rightarrow$ There exists a deterministic ℓ-periodic optimal policy

$$\pi^* = (\pi_0^*, \pi_1^*, \ldots, \pi_{\ell-1}^*)$$

and the optimal decisions are $a_t = \pi_{[t]}^*(x_t)$.

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Bellman operators: for all $i \in \{0, 1, \ldots, \ell - 1\}$,

- $T_{i,\pi} v = r_i + \gamma P_{i,\pi} v$
- $T_i v = \max_\pi T_{i,\pi} v$
- $\pi \in \mathcal{G}_i v \quad \Leftrightarrow \quad T_{i,\pi} v = T_i v$

For all t, for all periodic policy $\pi = (\pi_0, \pi_1, \ldots \pi_{\ell-1})$, the value when starting at time t satisfies

$$v_{[t],\pi} = T_{[t],\pi_{[t]}} T_{[t+1],\pi_{[t+1]}} \ldots T_{[t+\ell-1],\pi_{[t+\ell-1]}} v_{[t],\pi}$$

The optimal value function $v^* = (v^*_0, v^*_1, \ldots, v^*_{\ell-1})$ satisfies

$$\forall t, \quad v^*_{[t]} = T_{[t]} v^*_{[t+1]}$$

and an optimal policy $v^* = (\pi^*_0, \pi^*_1, \ldots, \pi^*_{\ell-1})$ is such that $\pi^*_{[t]} \in \mathcal{G}_{[t]} v^*_{[t+1]}$.

Dynamic Programming Algorithms

Store in memory:
$$\begin{aligned} \pi &= (\pi_0, \quad \pi_1, \quad \dots, \quad \pi_{\ell-2}, \quad \pi_{\ell-1}) \\ v &= (v_0, \quad v_1, \quad \dots, \quad v_{\ell-2}, \quad v_{\ell-1}) \end{aligned}$$

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$$\pi_{[-k]} \leftarrow \mathcal{G}_{[-k]} v_{[-k+1]}$$
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2. Periodic Markov Decision Processes
3. **Approximate Dynamic Programming**

Approximate Dynamic Programming

- $[(T_\pi)^m v](x)$ approximated by Monte-Carlo:

$$[(T_\pi)^m v](x) = \mathbb{E}\left[\sum_{t=0}^{m-1} \gamma^t r(x_t, a_t) + \gamma^m v(x_m) \middle| x_0 = x, \{\forall t, a_t = \pi(x_t)\} \right]$$

- "$v(\cdot) \leftarrow [Au](\cdot)$" approximated by regression:

$$\min_{v \in \mathcal{F} \subset \mathbb{R}^X} \sum_x \mu(x) |v(x) - [Au](x)|^p$$

- "$\pi(\cdot) \leftarrow [\mathcal{G} f](\cdot)$" approximated by (cost-sensitive) classification

$$\min_{\pi \in \Pi \subset A^X} \sum_x \mu(x) \left(\max_a [T_a f](x) - [T_\pi f](x) \right)$$

Approximate Algorithms (stationary)

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Theorem (Singh & Yee, 1994; Gordon, 1995; Bertsekas & Tsitsiklis, 1996; Scherrer *et al.* , 2012; Scherrer *et al.* , 2015)

Assume $\|\epsilon_k\|_\infty \leq \epsilon$. The *loss* due to running policy π_k instead of the optimal policy π_* satisfies

$$\limsup_{k\to\infty} \|v_{\pi_*} - v_{\pi_k}\|_\infty \leq \frac{2\gamma}{(1-\gamma)^2}\epsilon.$$

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The non-stationary trick

- The bigger the period ℓ, the better the bound $\frac{2\gamma}{(1-\gamma^\ell)(1-\gamma)}\epsilon$
- Any stationary MDP is a ℓ-periodic MDP for any $\ell \geq 1$
  (Scherrer & Lesner, 2012; Lesner & Scherrer, 2015; Perolat *et al.* , 2016)
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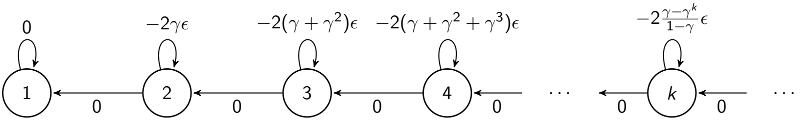
Tightness of the bound for $m = 0, \ell = 1$



	1	2	3	4	...
v_0	0	0	0	0	...
v_1	$-\epsilon$	ϵ	0	0	...
v_2	$-\gamma\epsilon$	$-\epsilon-\gamma\epsilon$	$\epsilon+\gamma\epsilon$	0	...
v_3	$-\gamma^2\epsilon$	$-\gamma^2\epsilon$	$-\epsilon-\gamma\epsilon-\gamma^2\epsilon$	$\epsilon+\gamma\epsilon+\gamma^2\epsilon$	...
...	...	...	...	...	...

State 2: $0+\gamma(-\epsilon) = -2\gamma\epsilon+\gamma\epsilon$
State 3: $0+\gamma(-\epsilon-\gamma\epsilon) = -2(\gamma+\gamma^2)\epsilon+\gamma(\epsilon+\gamma\epsilon)$

$$v_{\pi_k}(k) = \sum_{t=0}^{\infty}\gamma^t\left(-2\frac{\gamma-\gamma^k}{1-\gamma}\epsilon\right) = -2\frac{\gamma-\gamma^k}{(1-\gamma)^2}\epsilon \xrightarrow{k\to\infty} -\frac{2\gamma}{(1-\gamma)^2}\epsilon$$

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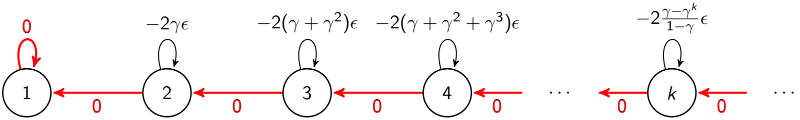

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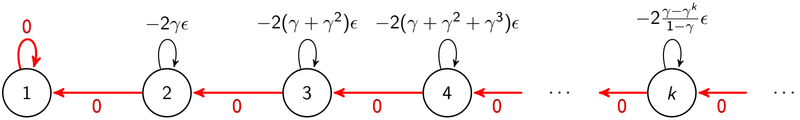

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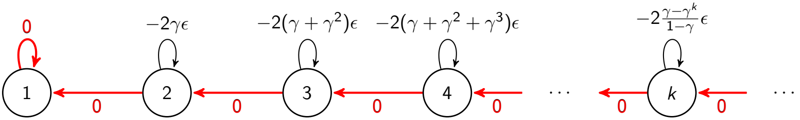
Tightness of the bound for $m = 0, \ell = 1$



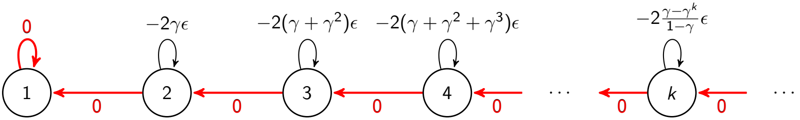
	1	2	3	4	...
v_0	0	0	0	0	...
v_1	$-\epsilon$	ϵ	0	0	...
v_2	$-\gamma\epsilon$	$-\epsilon-\gamma\epsilon$	$\epsilon+\gamma\epsilon$	0	...
v_3	$-\gamma^2\epsilon$	$-\gamma^2\epsilon$	$-\epsilon-\gamma\epsilon-\gamma^2\epsilon$	$\epsilon+\gamma\epsilon+\gamma^2\epsilon$	...
...	...	...	...	...	...

State 2: $0+\gamma(-\epsilon) = -2\gamma\epsilon+\gamma\epsilon$

State 3: $0+\gamma(-\epsilon-\gamma\epsilon) = -2(\gamma+\gamma^2)\epsilon+\gamma(\epsilon+\gamma\epsilon)$

$$v_{\pi_k}(k) = \sum_{t=0}^{\infty}\gamma^t\left(-2\frac{\gamma-\gamma^k}{1-\gamma}\epsilon\right) = -2\frac{\gamma-\gamma^k}{(1-\gamma)^2}\epsilon \xrightarrow{k\to\infty} -\frac{2\gamma}{(1-\gamma)^2}\epsilon$$

Tightness of the bound for $m = 0, \ell = 1$



	1	2	3	4	...
v_0	0	0	0	0	...
v_1	$-\epsilon$	ϵ	0	0	...
v_2	$-\gamma\epsilon$	$-\epsilon-\gamma\epsilon$	$\epsilon+\gamma\epsilon$	0	...
v_3	$-\gamma^2\epsilon$	$-\gamma^2\epsilon$	$-\epsilon-\gamma\epsilon-\gamma^2\epsilon$	$\epsilon+\gamma\epsilon+\gamma^2\epsilon$	...
...	...	...	...	...	...

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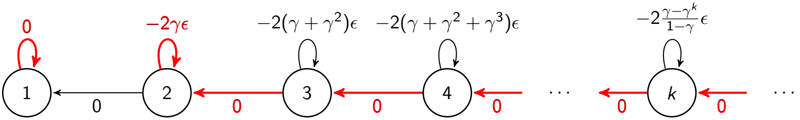
$$v_{\pi_k}(k) = \sum_{t=0}^{\infty}\gamma^t\left(-2\frac{\gamma-\gamma^k}{1-\gamma}\epsilon\right) = -2\frac{\gamma-\gamma^k}{(1-\gamma)^2}\epsilon \overset{k\to\infty}{\longrightarrow} -\frac{2\gamma}{(1-\gamma)^2}\epsilon$$

Tightness of the bound for $m = 0, \ell = 1$



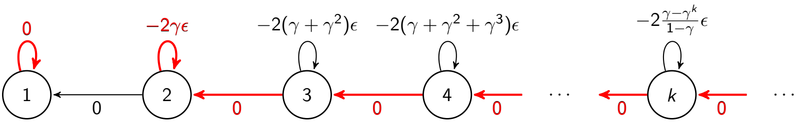
	1	2	3	4	...
v_0	0	0	0	0	...
v_1	$-\epsilon$	ϵ	0	0	...
v_2	$-\gamma\epsilon$	$-\epsilon-\gamma\epsilon$	$\epsilon+\gamma\epsilon$	0	...
v_3	$-\gamma^2\epsilon$	$-\gamma^2\epsilon$	$-\epsilon-\gamma\epsilon-\gamma^2\epsilon$	$\epsilon+\gamma\epsilon+\gamma^2\epsilon$	...
...	...	...	...	...	...

State 2: $0 + \gamma(-\epsilon) = -2\gamma\epsilon + \gamma\epsilon$

State 3: $0 + \gamma(-\epsilon - \gamma\epsilon) = -2(\gamma+\gamma^2)\epsilon + \gamma(\epsilon + \gamma\epsilon)$

$$v_{\pi_k}(k) = \sum_{t=0}^{\infty} \gamma^t \left(-2\frac{\gamma-\gamma^k}{1-\gamma}\epsilon\right) = -2\frac{\gamma-\gamma^k}{(1-\gamma)^2}\epsilon \xrightarrow{k\to\infty} -\frac{2\gamma}{(1-\gamma)^2}\epsilon$$

Tightness of the bound for $m = 0, \ell = 1$



	1	2	3	4	...
v_0	0	0	0	0	...
v_1	$-\epsilon$	ϵ	0	0	...
v_2	$-\gamma\epsilon$	$-\epsilon - \gamma\epsilon$	$\epsilon + \gamma\epsilon$	0	...
v_3	$-\gamma^2\epsilon$	$-\gamma^2\epsilon$	$-\epsilon - \gamma\epsilon - \gamma^2\epsilon$	$\epsilon + \gamma\epsilon + \gamma^2\epsilon$	...
...	...	...	...	...	...

State 2: $0 + \gamma(-\epsilon) = -2\gamma\epsilon + \gamma\epsilon$
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$$v_{\pi_k}(k) = \sum_{t=0}^{\infty} \gamma^t \left(-2\frac{\gamma - \gamma^k}{1-\gamma}\epsilon\right) = -2\frac{\gamma - \gamma^k}{(1-\gamma)^2}\epsilon \overset{k\to\infty}{\longrightarrow} -\frac{2\gamma}{(1-\gamma)^2}\epsilon$$

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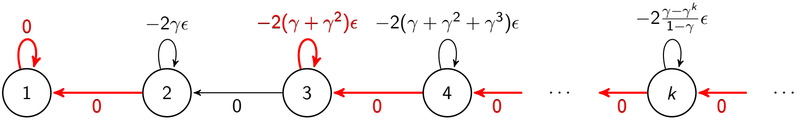

	1	2	3	4	. . .
v_0	0	0	0	0	. . .
v_1	$-\epsilon$	ϵ	0	0	. . .
v_2	$-\gamma\epsilon$	$-\epsilon - \gamma\epsilon$	$\epsilon + \gamma\epsilon$	0	. . .
v_3	$-\gamma^2\epsilon$	$-\gamma^2\epsilon$	$-\epsilon - \gamma\epsilon - \gamma^2\epsilon$	$\epsilon + \gamma\epsilon + \gamma^2\epsilon$	. . .
. . .	. . .	. . .	. . .	. . .	. . .

State 2: $0 + \gamma(-\epsilon) = -2\gamma\epsilon + \gamma\epsilon$
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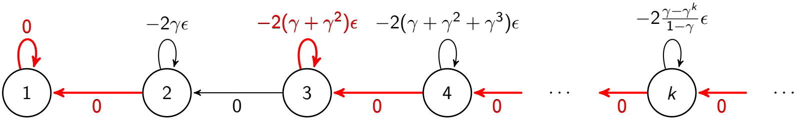
$$v_{\pi_k}(k) = \sum_{t=0}^{\infty} \gamma^t \left(-2\frac{\gamma - \gamma^k}{1-\gamma}\epsilon\right) = -2\frac{\gamma - \gamma^k}{(1-\gamma)^2}\epsilon \overset{k\to\infty}{\longrightarrow} -\frac{2\gamma}{(1-\gamma)^2}\epsilon$$

Tightness of the bound for $m = 0, \ell = 1$



	1	2	3	4	...
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v_1	$-\epsilon$	ϵ	0	0	...
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v_3	$-\gamma^2\epsilon$	$-\gamma^2\epsilon$	$-\epsilon-\gamma\epsilon-\gamma^2\epsilon$	$\epsilon+\gamma\epsilon+\gamma^2\epsilon$	...
...	...	...	...	...	...

State 2: $0+\gamma(-\epsilon) = -2\gamma\epsilon + \gamma\epsilon$
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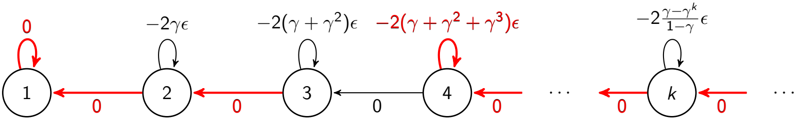

	1	2	3	4	...
v_0	0	0	0	0	...
v_1	$-\epsilon$	ϵ	0	0	...
v_2	$-\gamma\epsilon$	$-\epsilon-\gamma\epsilon$	$\epsilon+\gamma\epsilon$	0	...
v_3	$-\gamma^2\epsilon$	$-\gamma^2\epsilon$	$-\epsilon-\gamma\epsilon-\gamma^2\epsilon$	$\epsilon+\gamma\epsilon+\gamma^2\epsilon$	...
...	...	...	...	...	...

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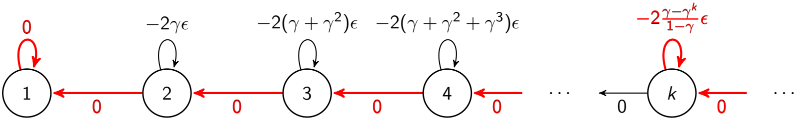
$$v_{\pi_k}(k) = \sum_{t=0}^{\infty}\gamma^t\left(-2\frac{\gamma-\gamma^k}{1-\gamma}\epsilon\right) = -2\frac{\gamma-\gamma^k}{(1-\gamma)^2}\epsilon \xrightarrow{k\to\infty} -\frac{2\gamma}{(1-\gamma)^2}\epsilon$$

Tightness of the bound for $m = 0, \ell = 1$



	1	2	3	4	…
v_0	0	0	0	0	…
v_1	$-\epsilon$	ϵ	0	0	…
v_2	$-\gamma\epsilon$	$-\epsilon-\gamma\epsilon$	$\epsilon+\gamma\epsilon$	0	…
v_3	$-\gamma^2\epsilon$	$-\gamma^2\epsilon$	$-\epsilon-\gamma\epsilon-\gamma^2\epsilon$	$\epsilon+\gamma\epsilon+\gamma^2\epsilon$	…
…	…	…	…	…	…

State 2: $0+\gamma(-\epsilon) = -2\gamma\epsilon + \gamma\epsilon$
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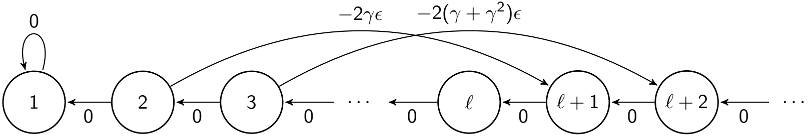
$$v_{\pi_k}(k) = \sum_{t=0}^{\infty} \gamma^t \left(-2\frac{\gamma-\gamma^k}{1-\gamma}\epsilon\right) = -2\frac{\gamma-\gamma^k}{(1-\gamma)^2}\epsilon \xrightarrow{k\to\infty} -\frac{2\gamma}{(1-\gamma)^2}\epsilon$$

Tightness of the bound (Lesner & Scherrer, 2015)



For any m and ℓ, ADP generates a sequence of policies $(\pi_{[-k]})_{k\geq 1}$ such that $\pi_{[-k]}$ acts optimally except in state k.
Thus, the resulting policy $\pi = (\pi_0, \ldots, \pi_{\ell-1})$ gets stuck in the loop

$$k, \quad k+\ell-1, \quad k+\ell-2, \quad k+1, \quad k, \quad \ldots$$

and therefore

$$v_{[-k],\pi}(k) = -\frac{2\gamma - \gamma^k}{(1-\gamma^\ell)(1-\gamma)}\epsilon.$$

Simulations

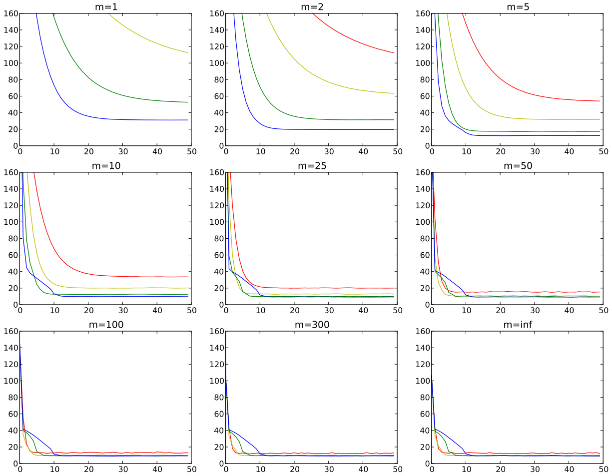


Figure: Average error of policy π per iteration k ADP. $\ell = 1$, $\ell = 2$, $\ell = 5$, $\ell = 10$.

Conclusion

This talk

- Periodic Markov Decision Processes
- The bigger the period, the better the (tight) performance guarantee
- (Stationary) MDPs are also periodic MDPs

Beyond deterministic stationary policies

- periodic deterministic policies
- probabilistic policies (Conservative Policy Iteration) (Kakade & Langford, 2002)

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Illustration of approximation on Tetris

1. Approximation architecture for the value and for the score (on which the policy is based)

$$
\begin{aligned}
f_\theta(x) = \theta_0 & \qquad \text{Constant} \\
+ \theta_1 h_1(x) + \theta_2 h_2(x) + \cdots + \theta_{10} h_{10}(x) & \qquad \text{column height} \\
+ \theta_{11} \Delta h_1(x) + \theta_{12} \Delta h_2(x) + \cdots + \theta_{19} \Delta h_9(x) & \qquad \text{height variation} \\
+ \theta_{20} \max_k h_k(x) & \qquad \text{max height} \\
+ \theta_{21} L(x) & \qquad \# \text{ holes}
\end{aligned}
$$

2. Sampling Scheme: play games